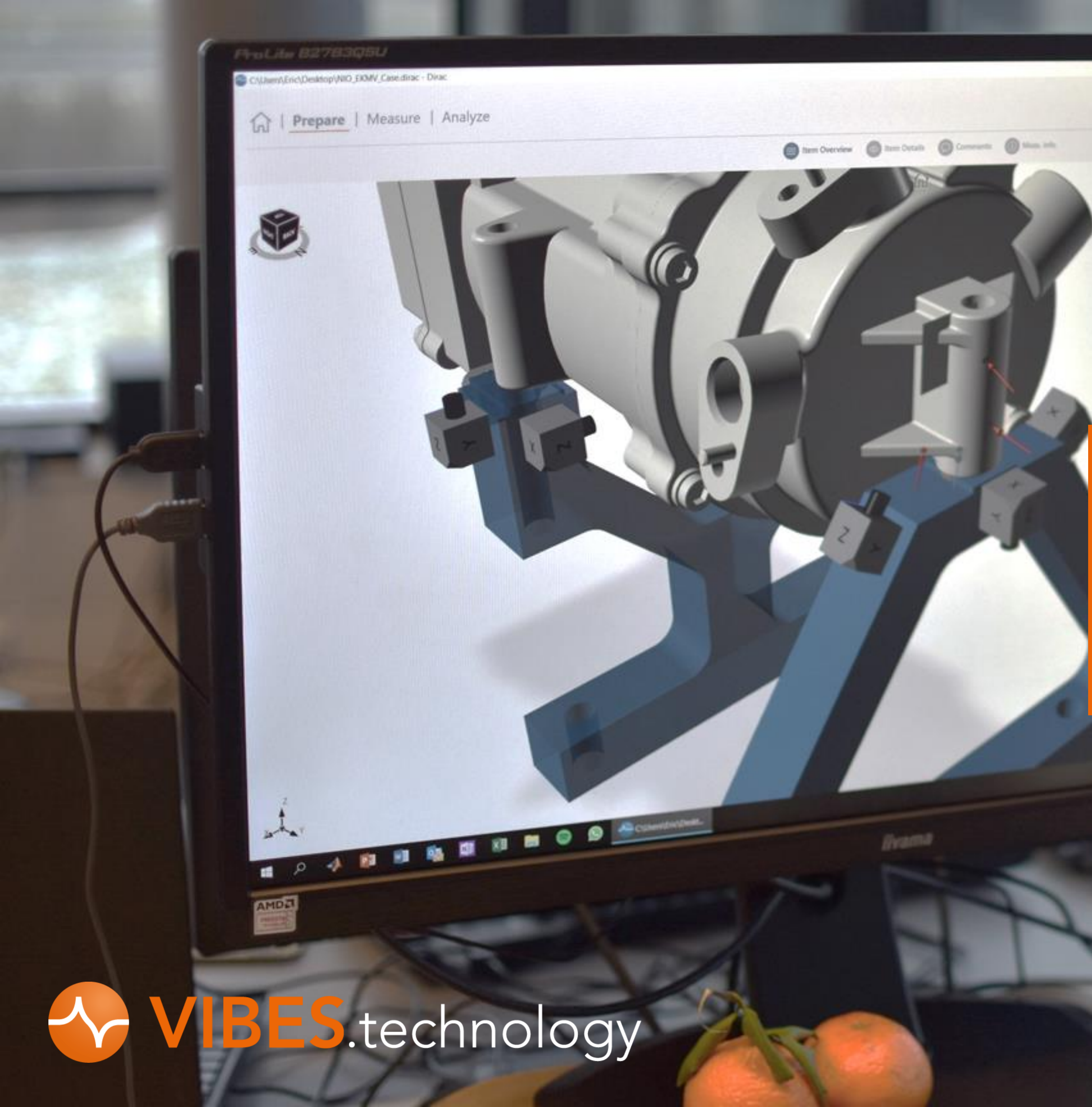

Tutorial on Experimental Substructuring with application to Joint Identification

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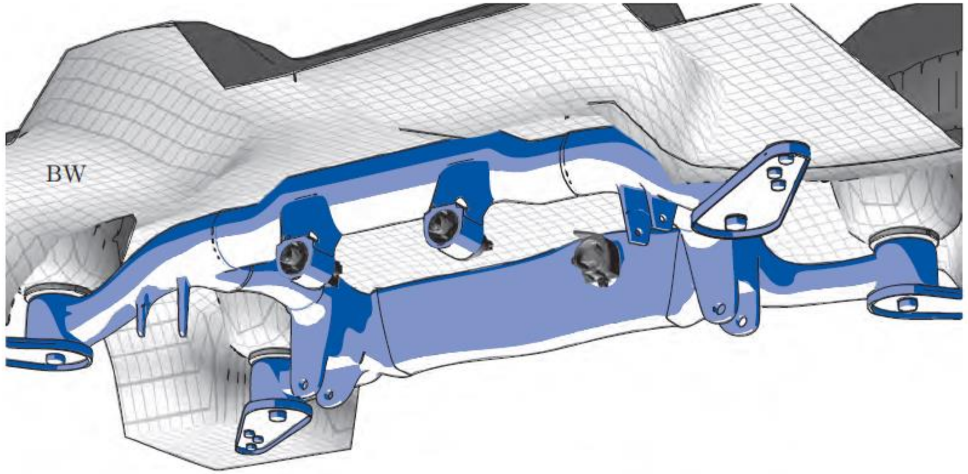
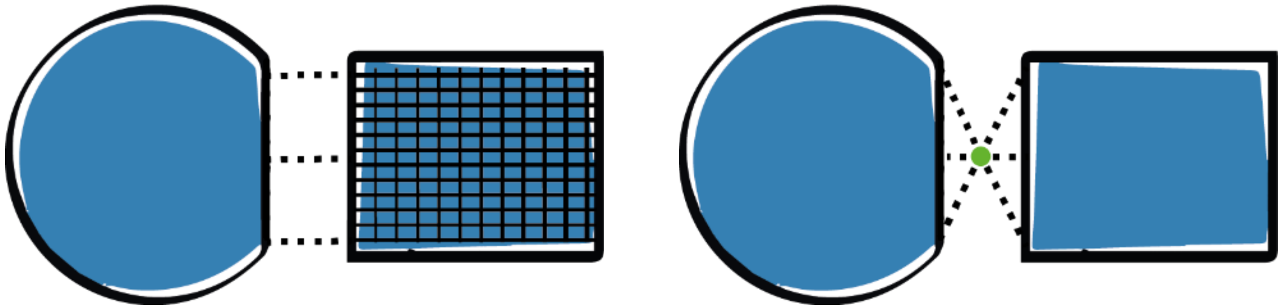
This tutorial will cover...

- **Substructuring Basics**
  - Coupling conditions: B and L-Matrix
  - Primal Assembly / CMS
  - Dual Assembly / LM-FBS
- **Practice of Dynamic Substructuring**
  - Virtual Point Transformation
  - Example: Experimental VP & FBS
- **Joint Identification**
  - Inverse substructuring
  - FBS decoupling
  - Example: Inverse substructuring of rubber mount

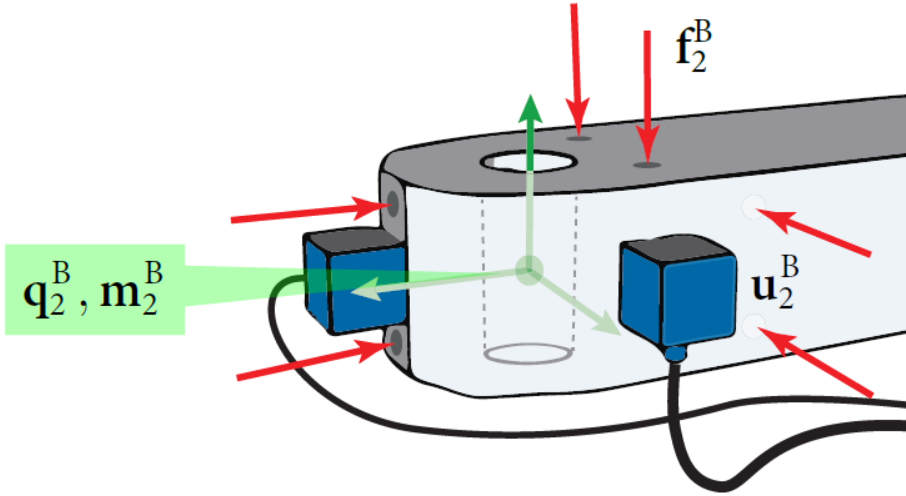
What is Dynamic Substructuring (DS)?

Substructuring is a “**way to see things in parts**” in order to simplify the dynamic analysis and concentrate on specific components in their assembled context.





DS allows for different model representations to be combined, such as numerical and experimental models, such that each substructure can be modelled in **the most cost-effective way**.



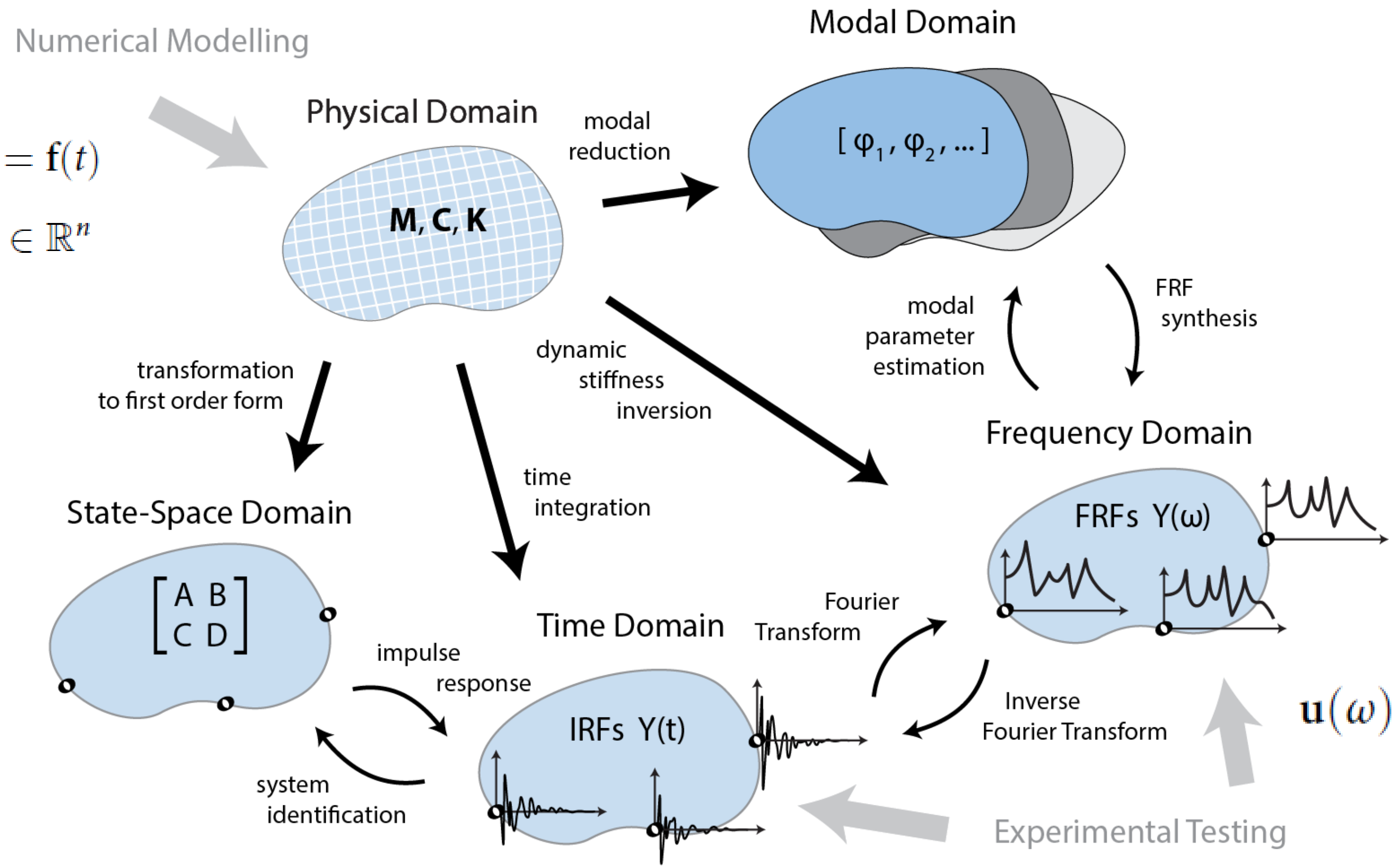
The key to successful (experimental) substructuring is not so much implementing the right equations, but more of correctly describing the dynamics at the **substructures’ interfaces.**

How does Dynamic Substructuring work?
The " three-field formulation"

1. **Equations of motion** representing subsystem dynamics

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t)$$

$$\mathbf{u} \in \mathbb{R}^n$$



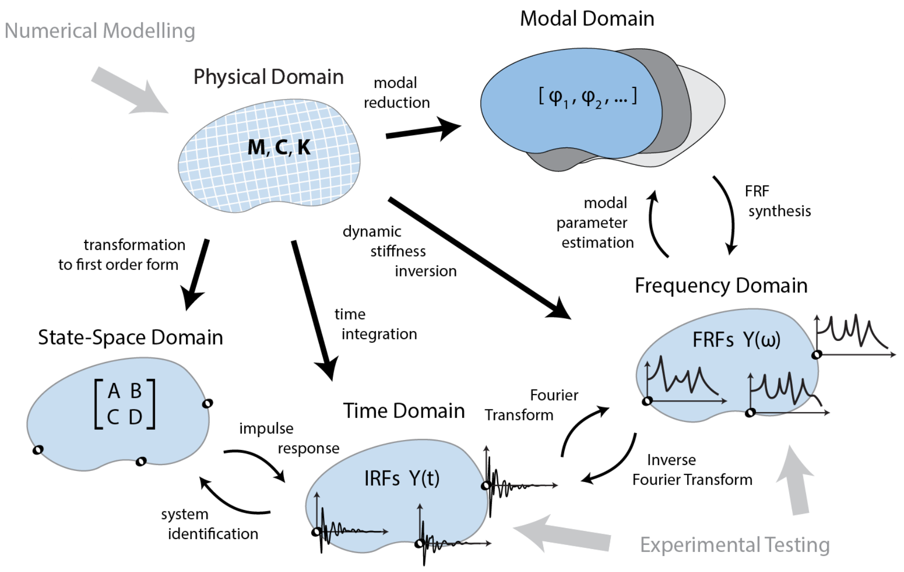
$$\mathbf{u}(\omega) = \mathbf{Y}(\omega)\mathbf{f}(\omega)$$

How does Dynamic Substructuring work?

The “ three-field formulation”

1. **Equations of motion** representing subsystem dynamics

numerical, frequency-based, modal-based, …

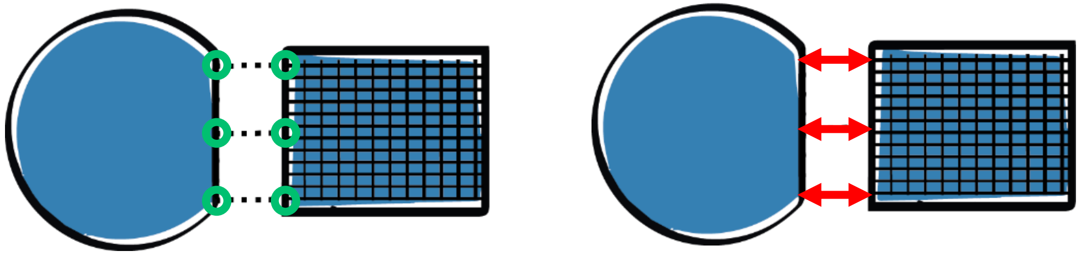


2. **Coordinate compatibility** at coupling DoFs

keeping the subsystems connected

3. **Force equilibrium** between coupling DoFs

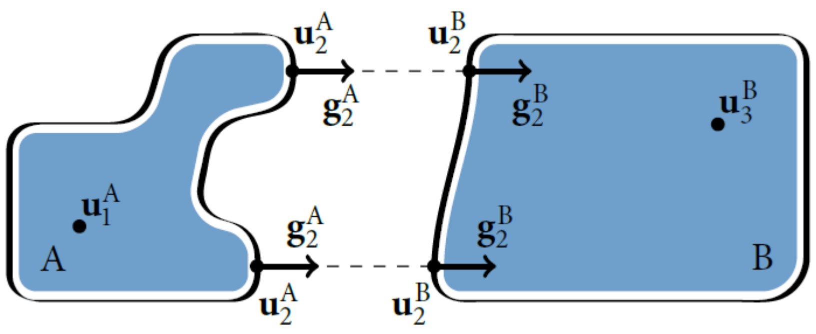
“action == minus reaction” connecting forces



SUBSTRUCTURE ASSEMBLY

The “three-field formulation”

1. Substructure’s equations of motion (e.g. physical domain):



$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f} + \mathbf{g}$$

$$\mathbf{u} \triangleq \begin{bmatrix} \mathbf{u}_1^A \\ \mathbf{u}_2^A \\ \mathbf{u}_2^B \\ \mathbf{u}_3^B \end{bmatrix}; \quad \mathbf{f} \triangleq \begin{bmatrix} \mathbf{f}_1^A \\ \mathbf{f}_2^A \\ \mathbf{f}_2^B \\ \mathbf{f}_3^B \end{bmatrix}; \quad \mathbf{g} \triangleq \begin{bmatrix} 0 \\ \mathbf{g}_2^A \\ \mathbf{g}_2^B \\ 0 \end{bmatrix}; \quad \mathbf{u}, \mathbf{f}, \mathbf{g} \in \mathbb{R}^n$$

displacements — applied forces — connecting forces

2. Coordinate compatibility:

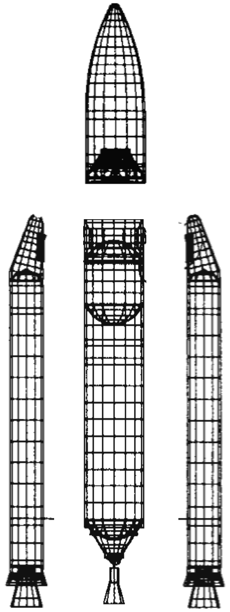
$$\mathbf{u}_2^A = \mathbf{u}_2^B \quad \Leftrightarrow \quad \mathbf{u}_2^B - \mathbf{u}_2^A = 0$$

3. Force equilibrium:

$$\mathbf{g}_2^A = -\mathbf{g}_2^B \quad \Leftrightarrow \quad \mathbf{g}_2^A + \mathbf{g}_2^B = 0$$

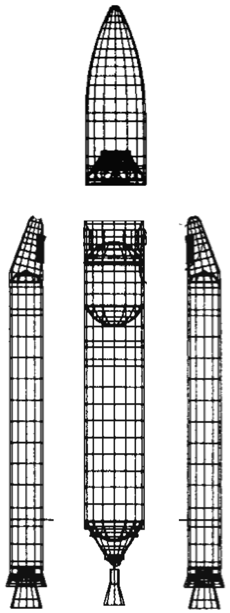
Fine to do this by hand for just 2, but how about doing this for n substructures?

$$\boldsymbol{M} \triangleq \text{diag}\left(\boldsymbol{M}^{(1)}, \dots, \boldsymbol{M}^{(n)} \right)$$
$$\boldsymbol{C} \triangleq \text{diag}\left(\boldsymbol{C}^{(1)}, \dots, \boldsymbol{C}^{(n)} \right)$$
$$\boldsymbol{K} \triangleq \text{diag}\left(\boldsymbol{K}^{(1)}, \dots, \boldsymbol{K}^{(n)} \right)$$



SUBSTRUCTURE ASSEMBLY

Two strategies: primal and dual assembly



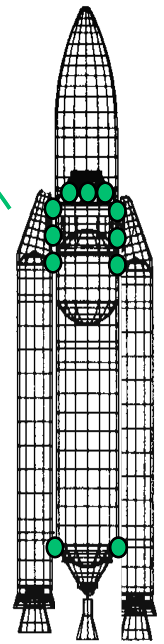
Primal assembly

1. ~~Coordinate compatibility on $\boldsymbol{u}$~~

   ➔ *satisfied a-priori by introducing generalized DoFs* $\boldsymbol{q}$

2. Force equilibrium on $\boldsymbol{g}$

$\boldsymbol{q_2}$ to substitute for the connecting DoFs $\boldsymbol{u_2}$

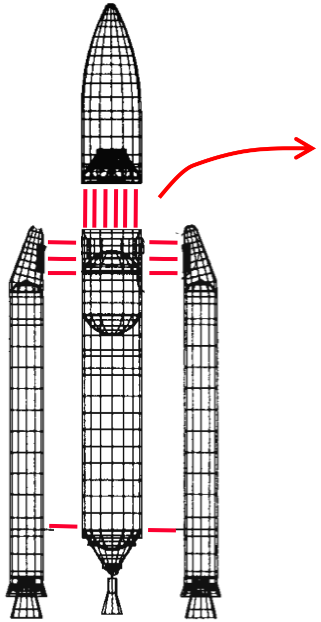


$$u_2^A = u_2^B \quad \Leftrightarrow \quad \begin{cases} u_2^A = q_2 \\ u_2^B = q_2 \end{cases}$$

Dual assembly

1. Coordinate compatibility on $\boldsymbol{u}$
2. ~~Force equilibrium on $\boldsymbol{g}$~~

   ➔ *satisfied a-priori by introducing Lagrange Multipliers* $\boldsymbol{\lambda}$



$\boldsymbol{\lambda}$ to substitute for the connecting forces $\boldsymbol{g_2}$

$$g_2^A = -g_2^B \quad \Leftrightarrow \quad \begin{cases} g_2^A = \lambda \\ g_2^B = -\lambda \end{cases}$$

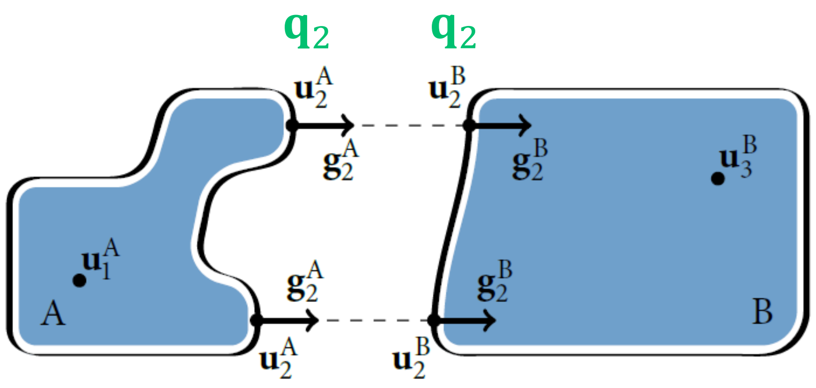
SUBSTRUCTURE ASSEMBLY

Primal assembly of system matrices using L-matrix

Primal assembly: use the **L** matrix to write both conditions

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f} + \mathbf{g}$$

$$\hat{\mathbf{M}}\ddot{\mathbf{q}}(t) + \hat{\mathbf{C}}\dot{\mathbf{q}}(t) + \hat{\mathbf{K}}\mathbf{q}(t) = \mathbf{p}(t) \quad \text{with} \quad \mathbf{p} = \mathbf{L}^T\mathbf{f}, \quad \begin{cases} \hat{\mathbf{M}} = \mathbf{L}^T\mathbf{M}\mathbf{L} \\ \hat{\mathbf{C}} = \mathbf{L}^T\mathbf{C}\mathbf{L} \\ \hat{\mathbf{K}} = \mathbf{L}^T\mathbf{K}\mathbf{L} \end{cases}$$



#DoF decreased by m coupling conditions!

Coordinate compatibility:

$$\mathbf{u}_2^A = \mathbf{u}_2^B \quad \Leftrightarrow \quad \begin{cases} \mathbf{u}_2^A = \mathbf{q}_2 \\ \mathbf{u}_2^B = \mathbf{q}_2 \end{cases}$$

(substitution)

$$\mathbf{u} = \mathbf{L}\mathbf{q} \quad \Longrightarrow \quad \begin{cases} \mathbf{u}_1^A = \mathbf{q}_1 \\ \mathbf{u}_2^A = \mathbf{q}_2 \\ \mathbf{u}_2^B = \mathbf{q}_2 \\ \mathbf{u}_3^B = \mathbf{q}_3 \end{cases} \quad \text{with} \quad \mathbf{L} \triangleq \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

Force equilibrium:

$$\mathbf{g}_2^A = -\mathbf{g}_2^B \quad \Leftrightarrow \quad \mathbf{g}_2^A + \mathbf{g}_2^B = \mathbf{0}$$

(projection)

$$\mathbf{L}^T\mathbf{g} = \mathbf{0} \quad \Longrightarrow \quad \begin{cases} \mathbf{g}_1^A = \mathbf{0} \\ \mathbf{g}_2^A + \mathbf{g}_2^B = \mathbf{0} \\ \mathbf{g}_3^B = \mathbf{0} \end{cases}$$

connecting forces vanish in primal assembly when pre-multiplying Equations of Motions with $\mathbf{L}^T$

SUBSTRUCTURE ASSEMBLY

Dual assembly of FRFs using B-matrix ➔ LM-FBS

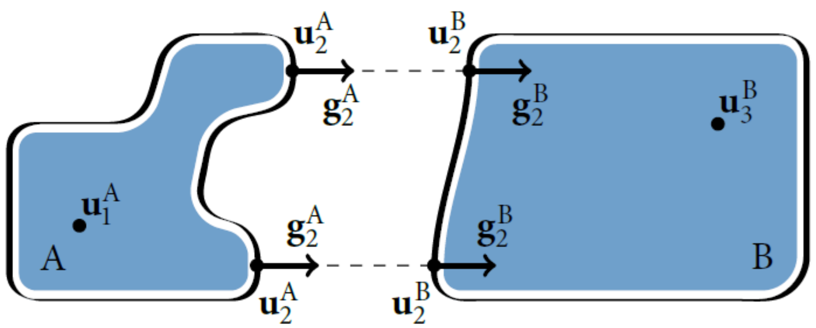
Dual assembly: use the B-matrix to write both conditions:

$$\mathbf{u} = \mathbf{Y}(\mathbf{f} + \mathbf{g}) \quad = \mathbf{Y}\mathbf{f} - \mathbf{Y}\mathbf{B}^T\boldsymbol{\lambda}$$

$$\mathbf{B}\mathbf{u} = \mathbf{0} \qquad \Rightarrow \quad \mathbf{B}\mathbf{u} = \mathbf{B}\mathbf{Y}\mathbf{f} - \mathbf{B}\mathbf{Y}\mathbf{B}^T\boldsymbol{\lambda} = \mathbf{0}$$

$$\boldsymbol{\lambda} = (\mathbf{B}\mathbf{Y}\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{Y}\mathbf{f}$$

$$\mathbf{u}^{\text{coupled}} = \mathbf{Y}\mathbf{f} - \mathbf{Y}\mathbf{B}^T(\mathbf{B}\mathbf{Y}\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{Y}\mathbf{f}$$



#DoF increased by m coupling conditions!

Coordinate compatibility:

(add. equation)

$$\mathbf{u}_2^A = \mathbf{u}_2^B \quad \Leftrightarrow \quad \mathbf{u}_2^B - \mathbf{u}_2^A = \mathbf{0} \qquad \mathbf{B}\mathbf{u} = \mathbf{0} \quad \Longrightarrow \quad \mathbf{u}_2^B - \mathbf{u}_2^A = \mathbf{0} \quad \text{with} \quad \mathbf{B} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} \end{bmatrix}$$

Force equilibrium:

(substitution)

$$\mathbf{g}_2^A = -\mathbf{g}_2^B \quad \Leftrightarrow \quad \begin{cases} \mathbf{g}_2^A = \boldsymbol{\lambda} \\ \mathbf{g}_2^B = -\boldsymbol{\lambda} \end{cases} \qquad \mathbf{g} = -\mathbf{B}^T\boldsymbol{\lambda} \quad \Longrightarrow \quad \begin{cases} \mathbf{g}_1^A = \mathbf{0} \\ \mathbf{g}_2^A = \boldsymbol{\lambda} \\ \mathbf{g}_2^B = -\boldsymbol{\lambda} \\ \mathbf{g}_3^B = \mathbf{0} \end{cases}$$

SUBSTRUCTURE ASSEMBLY

Interpretation of dual assembly through LM-FBS equation

$$\mathbf{u}^{\text{coupled}} = \mathbf{Yf} - \mathbf{YB}^T(\mathbf{BYB}^T)^{-1}\mathbf{BYf}$$

uncoupled response, creating a *gap between the interface DoFs*

$$\mathbf{u}^{\text{coupled}} = \mathbf{Yf} - \mathbf{YB}^T(\mathbf{BYB}^T)^{-1}\mathbf{BYf}$$

dynamic stiffness "felt" between the interface DoFs

$$\mathbf{u}^{\text{coupled}} = \mathbf{Yf} - \mathbf{YB}^T(\mathbf{BYB}^T)^{-1}\mathbf{BYf}$$

Interface forces, *distributed* back to the coupling DoFs

$$\mathbf{u}^{\text{coupled}} = \left(\mathbf{Y} - \mathbf{YB}^T(\mathbf{BYB}^T)^{-1}\mathbf{BY}\right)\mathbf{f}$$

$$\mathbf{Y}^{\text{coupled}} = \mathbf{Y} - \mathbf{YB}^T(\mathbf{BYB}^T)^{-1}\mathbf{BY}$$

Can be "weakened" to do compliant coupling

Coupled FRF, obtained only from subsystem FRFs and a Boolean Matrix

SUBSTRUCTURE ASSEMBLY

Summary of primal and dual assembly

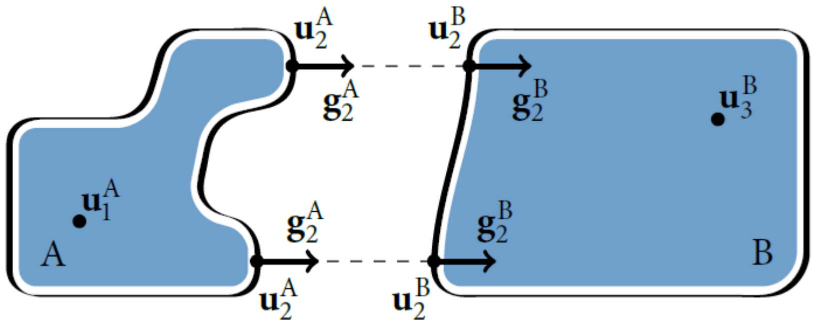
Primal Assembly:

- Compatibility enforced a-priori using $\mathbf{u} = \mathbf{L}\mathbf{q}$
- Total number of DoFs decreases to $n - m$
- Great for numerical substructuring ➔ **Component Mode Synthesis (CMS)**

Dual Assembly:

- Equilibrium enforced a-priori using $\mathbf{g} = -\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}$
- Total number of DoFs increases to $n + m$
- Great for experimental substructuring ➔ **LM-FBS**

Remark: B and L are each other's null-spaces!



Coordinate compatibility:

$$\mathbf{B}\mathbf{u} = \mathbf{0} \qquad \mathbf{B} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{I} & \mathbf{0} \end{bmatrix}$$

Force equilibrium:

$$\mathbf{L}^T\mathbf{g} = \mathbf{0} \qquad \mathbf{L} \triangleq \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

Primal Impedance Assembly:

$$\mathbf{u} = \mathbf{L}\mathbf{q} \qquad \mathbf{L}^T\mathbf{g} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \mathbf{f}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{11}^A & \mathbf{Z}_{12}^A & 0 \\ \mathbf{Z}_{21}^A & \mathbf{Z}_{22}^A + \mathbf{Z}_{22}^B & \mathbf{Z}_{23}^B \\ 0 & \mathbf{Z}_{32}^B & \mathbf{Z}_{33}^B \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix}$$

Dual Admittance Assembly:

$$\mathbf{B}\mathbf{u} = \mathbf{0} \qquad \mathbf{g} = -\mathbf{B}^T\boldsymbol{\lambda}$$

$$\begin{bmatrix} \mathbf{u}_1^A \\ \mathbf{u}_2^A \\ \mathbf{u}_2^B \\ \mathbf{u}_3^B \end{bmatrix} = \begin{bmatrix} \mathbf{Y}_{11}^A & \mathbf{Y}_{12}^A & 0 & 0 \\ \mathbf{Y}_{21}^A & \mathbf{Y}_{22}^A & 0 & 0 \\ 0 & 0 & \mathbf{Y}_{22}^B & \mathbf{Y}_{23}^B \\ 0 & 0 & \mathbf{Y}_{32}^B & \mathbf{Y}_{33}^B \end{bmatrix} \begin{bmatrix} \mathbf{f}_1^A \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{g}_2^A \\ \mathbf{g}_2^B \\ 0 \end{bmatrix}$$

This tutorial will cover...

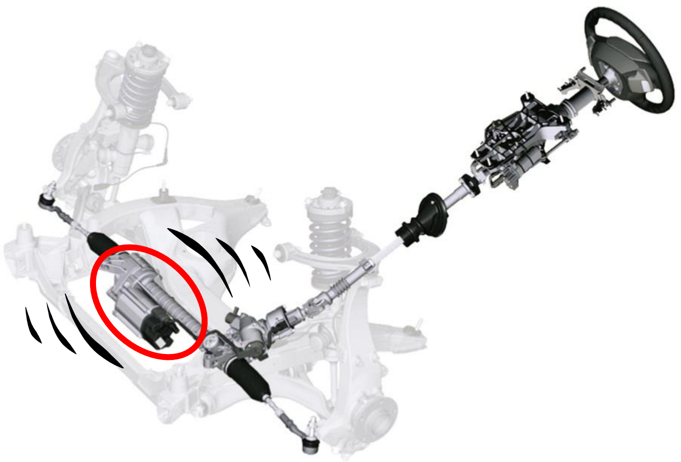
- **Substructuring Basics**
  - Coupling conditions: B and L-Matrix
  - Primal Assembly / CMS
  - Dual Assembly / LM-FBS

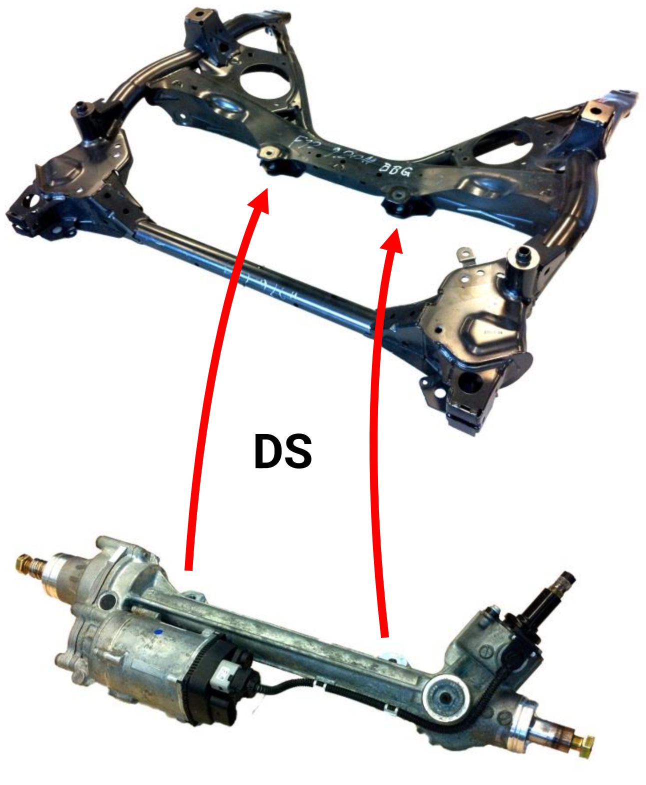
- **Practice of Dynamic Substructuring**
  - Virtual Point Transformation
  - Example: Experimental VP & FBS

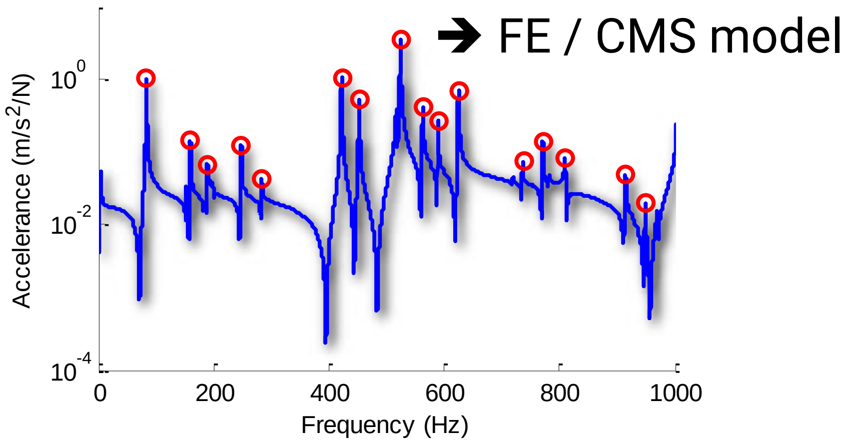
- **Joint Identification**
  - Inverse substructuring
  - FBS decoupling
  - Example: Inverse substructuring of rubber mount

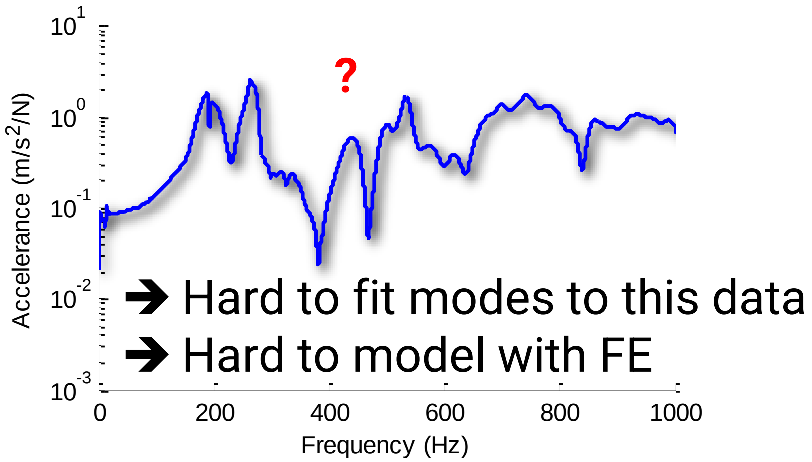
EXPERIMENTAL DYNAMIC SUBSTRUCTURING

Why do we need dynamic substructuring with measured components?





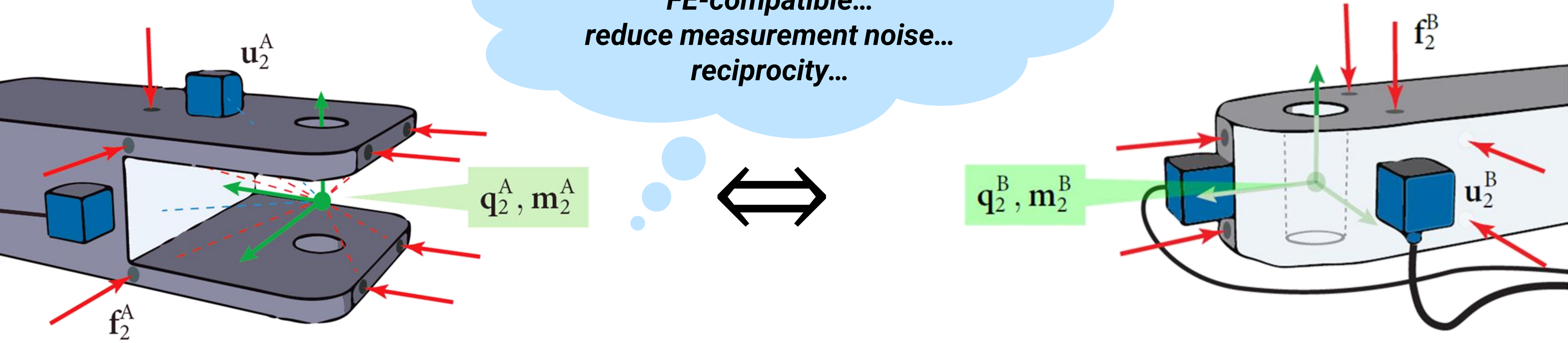




EXPERIMENTAL DYNAMIC SUBSTRUCTURING

Concept of virtual points

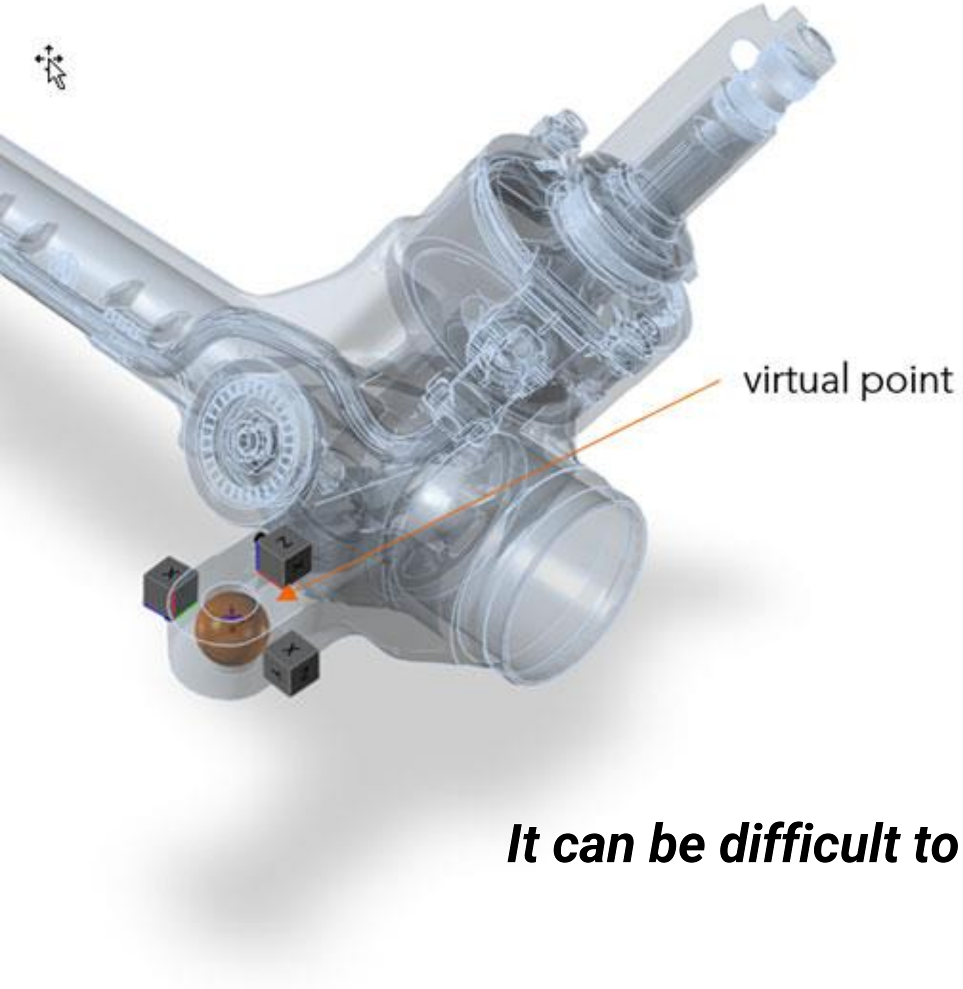
translation & rotation DoF…
drive-points…
FE-compatible…
reduce measurement noise…
reciprocity…



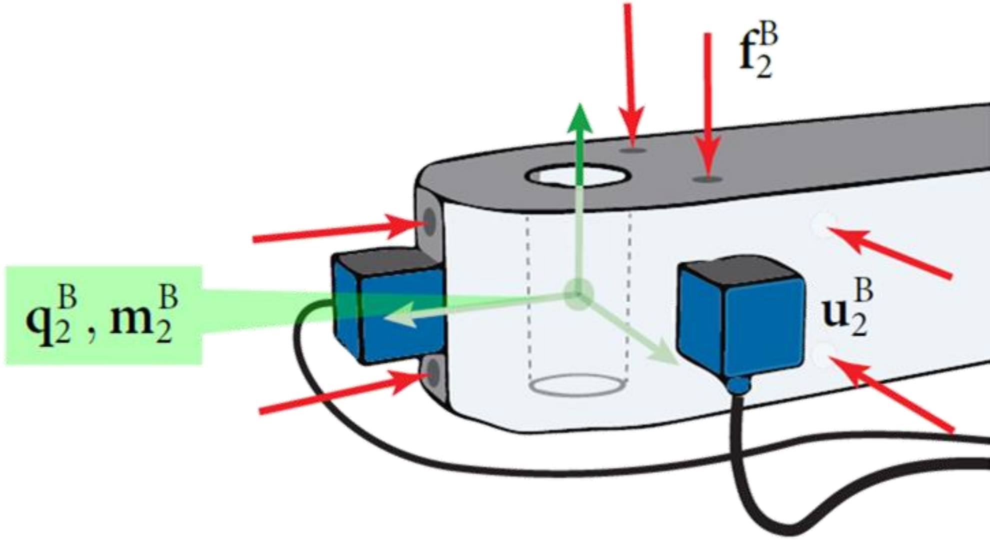
It can be difficult to gather measurements at interface degrees of freedom?

EXPERIMENTAL DYNAMIC SUBSTRUCTURING

Concept of virtual points



$\Leftrightarrow$



It can be difficult to gather measurements at interface degrees of freedom?

➔ It's easy, using Virtual Points!

Virtual Point Transformation

Projection of measured displacements on interface displacement modes (IDM)

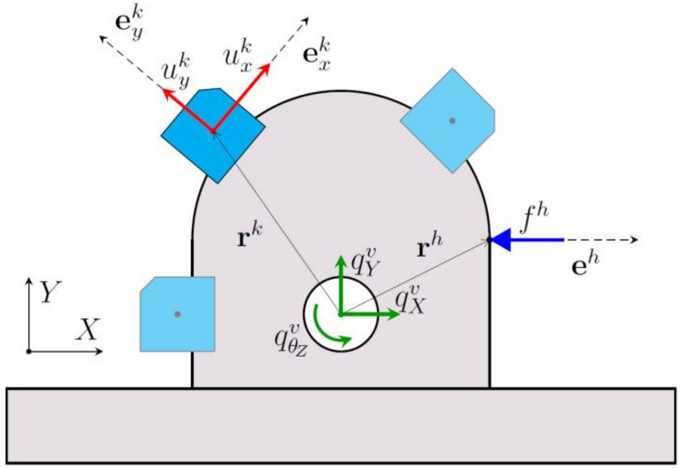
$$\mathbf{u} = \mathbf{R}\mathbf{q} + \boldsymbol{\mu}$$

***n* measurements (typically *n* = 9)**
i.e. accelerometer channels

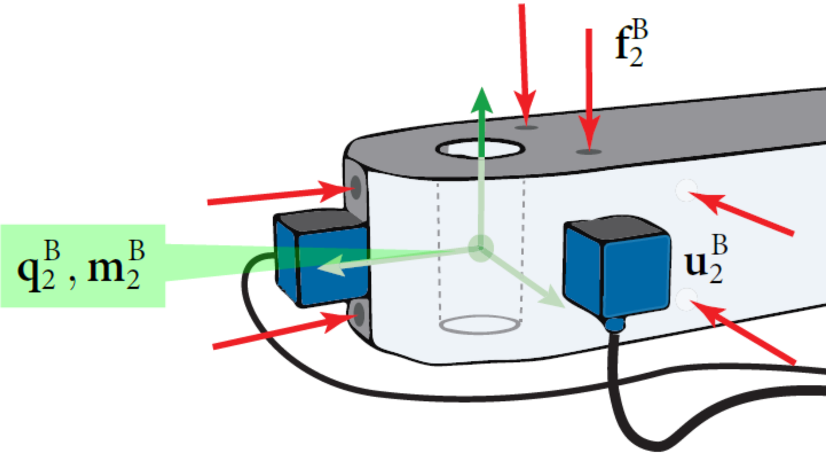
6 virtual DoFs
i.e. translations and rotations

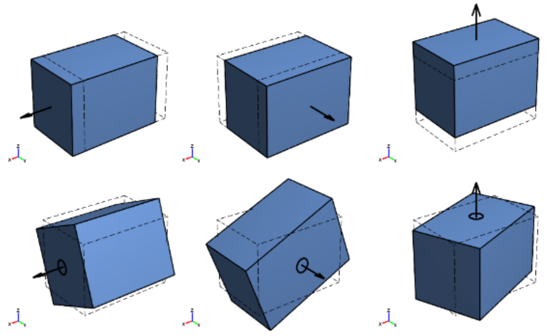
Remainder:
(i.e. noise, flexibility)

Reduction basis (IDMs)
I.e. kinematic relation w.r.t. the virtual point



Typical instrumentation of a coupling point





The 6 degrees of freedom in the virtual point modes (**R**)

Simple rigid transformation for one triaxial accelerometer:

$$\begin{bmatrix} u_X^k \\ u_Y^k \\ u_Z^k \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & r_Z^k & -r_Y^k \\ 0 & 1 & 0 & -r_Z^k & 0 & r_X^k \\ 0 & 0 & 1 & r_Y^k & -r_X^k & 0 \end{array}\right] q^v + \begin{bmatrix} \mu_X^k \\ \mu_Y^k \\ \mu_Z^k \end{bmatrix}$$

VP translations **VP rotations**

extend for second, third accelerometer…

$$\mathbf{q} = \left(\mathbf{R}^{\mathrm{T}}\mathbf{R}\right)^{-1}\mathbf{R}^{\mathrm{T}}\mathbf{u} = \mathbf{T}\mathbf{u}$$

T is pseudo-inverse of **R**

“Least-square fit of u”

Virtual Point Transformation

Projection of measured forces on interface displacement modes (IDM)

6 virtual DoFs
i.e. forces and moments in virtual point

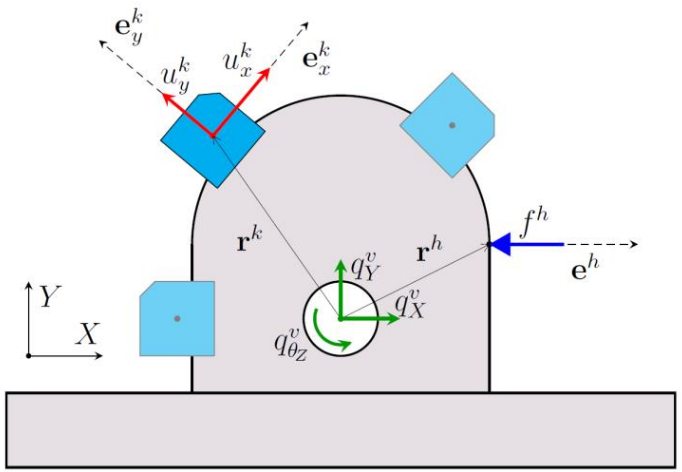
$$\mathbf{m} = \mathbf{R}_f^T \mathbf{f}$$

***n* force inputs**
i.e. hammer impacts or shaker positions

Reduction basis (IDMs)
I.e. kinematic relation w.r.t. the virtual point

$$\mathbf{m} = \begin{bmatrix} \mathbf{m}_t \\ \mathbf{m}_\theta \end{bmatrix} = \underbrace{\begin{bmatrix} {}_I\mathbf{e}^h \\ {}_I\mathbf{r}^h \times {}_I\mathbf{e}^h \end{bmatrix}}_{\text{projection matrix}} f^h$$

6×1

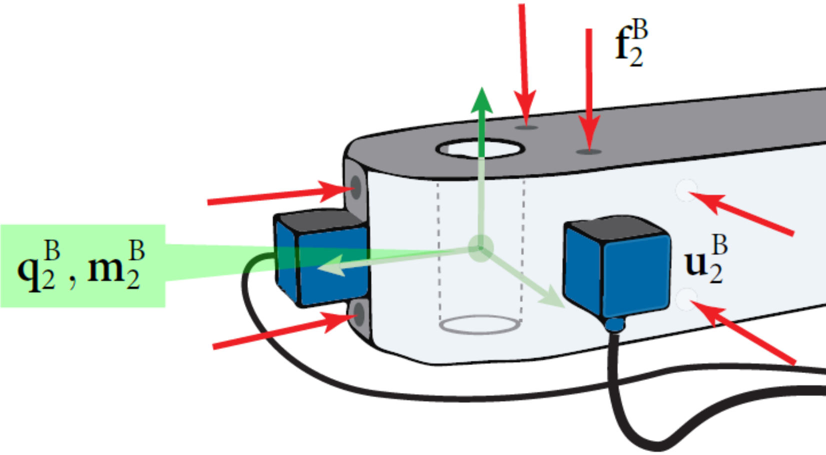


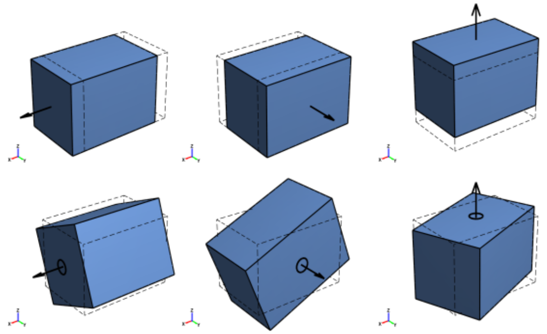
Typical instrumentation of a coupling point

$$\tilde{\boldsymbol{f}} = \boldsymbol{R}\left(\mathbf{R}^{\mathrm{T}}\mathbf{R}\right)^{-1}\mathbf{m} = \boldsymbol{T}^{\mathrm{T}}\mathbf{m}$$

$\mathbf{T}$ is pseudo-inverse of $\mathbf{R}$

"Constrained minimization of f"





The 6 degrees of freedom in the virtual point modes (**R**)

Virtual Point Transformation

Virtual point FRFs, practical benefits & quality criteria

Kinematic relation:

- Measured displacements: $\mathbf{u} = \mathbf{R_u q} + \boldsymbol{\mu}$
- Measured forces: $\mathbf{m} = \mathbf{R}_f{}^{\mathrm{T}}\mathbf{f}$

Coordinate transformation:

- VP displacements: $\mathbf{q} = \left(\mathbf{R}^{\mathrm{T}}\mathbf{R}\right)^{-1}\mathbf{R}^{\mathrm{T}}\mathbf{u} = \mathbf{T}\mathbf{u}$
- VP forces: $\mathbf{f} = \mathbf{R}\left(\mathbf{R}^{\mathrm{T}}\mathbf{R}\right)^{-1}\mathbf{m} = \mathbf{T}^{\mathrm{T}}\mathbf{m}$

Frequency response functions:

- Measured FRF: $\mathbf{u} = \mathbf{Y}(\boldsymbol{\omega})\mathbf{f}$
- Virtual point FRF: $\mathbf{q} = \mathbf{T}\mathbf{Y}(\boldsymbol{\omega})\mathbf{T}^{\mathrm{T}}\mathbf{m} = \mathbf{Y}_{\mathrm{qm}}(\boldsymbol{\omega})\mathbf{m}$
- $\mathbf{Y}_{\mathrm{qm}}(\omega)$ contains the 6-DoF FRFs at the virtual points estimated from $\mathbf{Y}(\omega)$

Practical benefits:

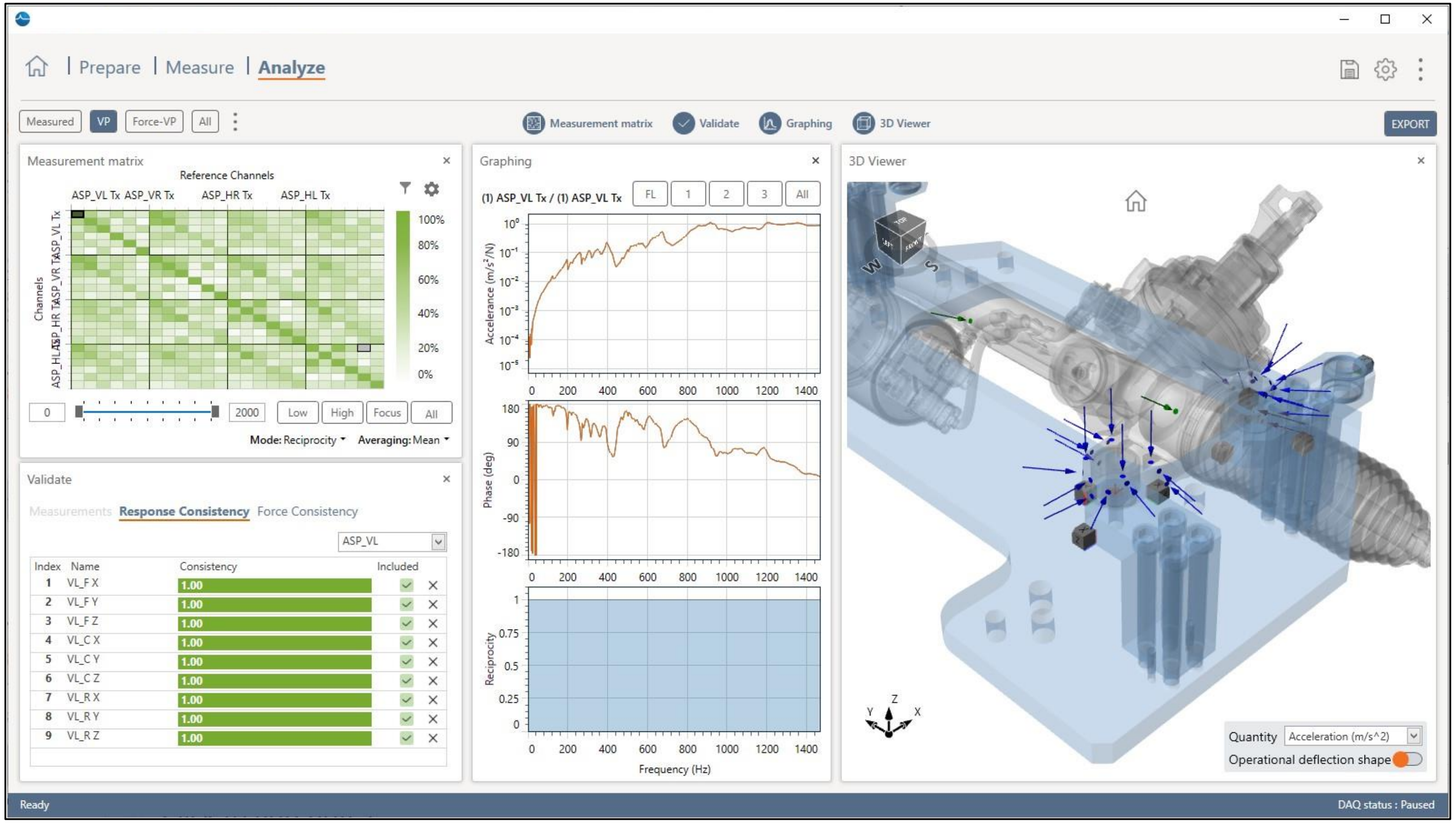
- Transformations $\mathbf{R_u}$ and $\mathbf{R_f}$ are <u>independent</u>! Therefore no excitations on the sensors are necessary
- Transformation can be performed on left ($\mathbf{u}$), right ($\mathbf{f}$) or both sides (useful for e.g. TPA)

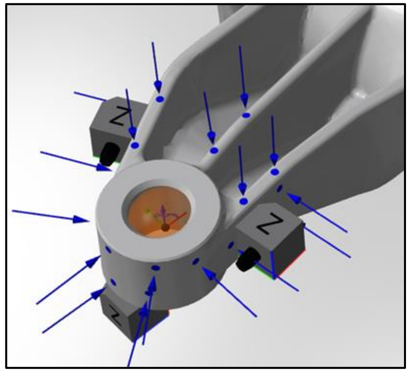
Quality criteria:

- *Consistency*: By comparing the ‘filtered’ displacements, one can estimate how ‘rigid’ the transformation is:
  $\mathbf{u} = \mathbf{Rq} + \boldsymbol{\mu} = \tilde{\mathbf{u}} + \boldsymbol{\mu} \;\rightarrow\; \tilde{\mathbf{u}} = \mathbf{RTu}$
- *Reciprocity*: as the virtual point FRF has collocated DoFs for forces and responses, strict reciprocity is required for the off-diagonal FRFs
- *Passivity*: for the diagonal (driving-point) FRFs, the phase is bound between 0 and 180 degrees.

Virtual Point Transformation

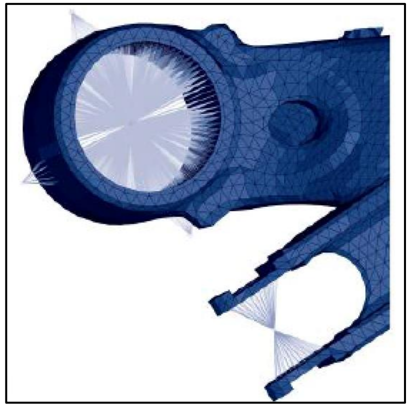
DIRAC for high-quality test-based modelling with VP





Test-based model (VP)

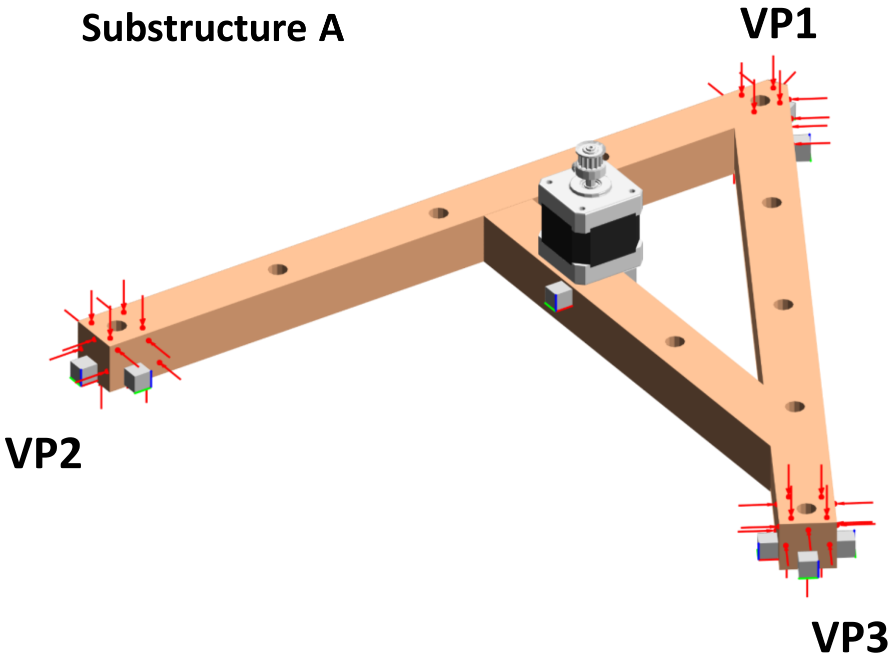
Numerical model (RBE3 / RBE2 element)

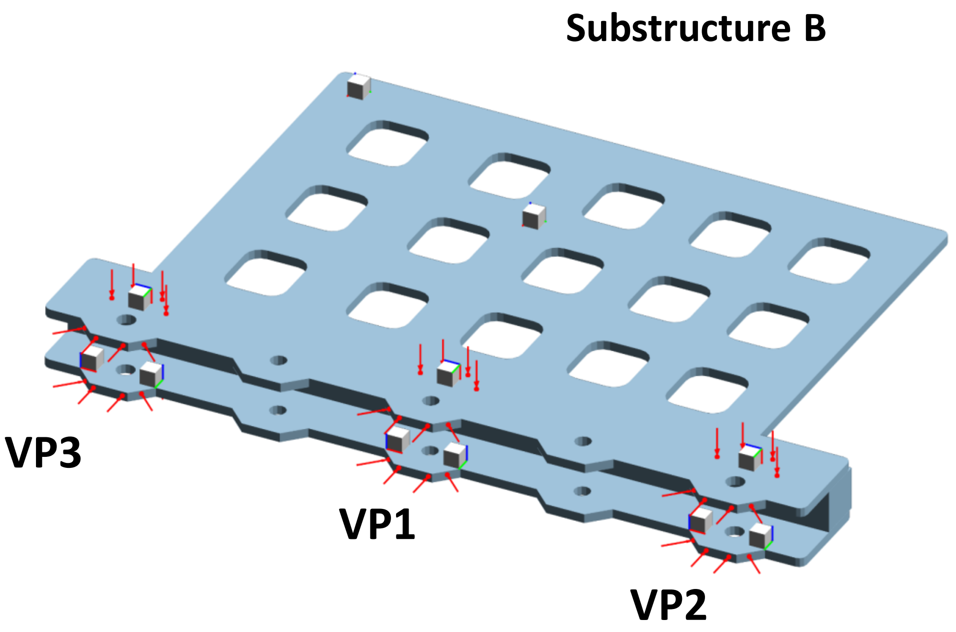


Examples

Substructure coupling using 2 experimentally modelled components

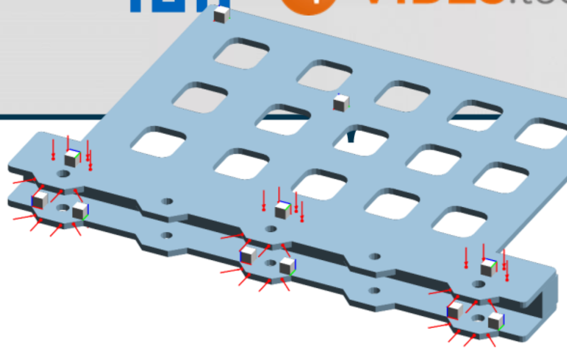
- Substructure A + B = Assembly AB
- Two ‘virtual coupling points, each with 6 DoFs



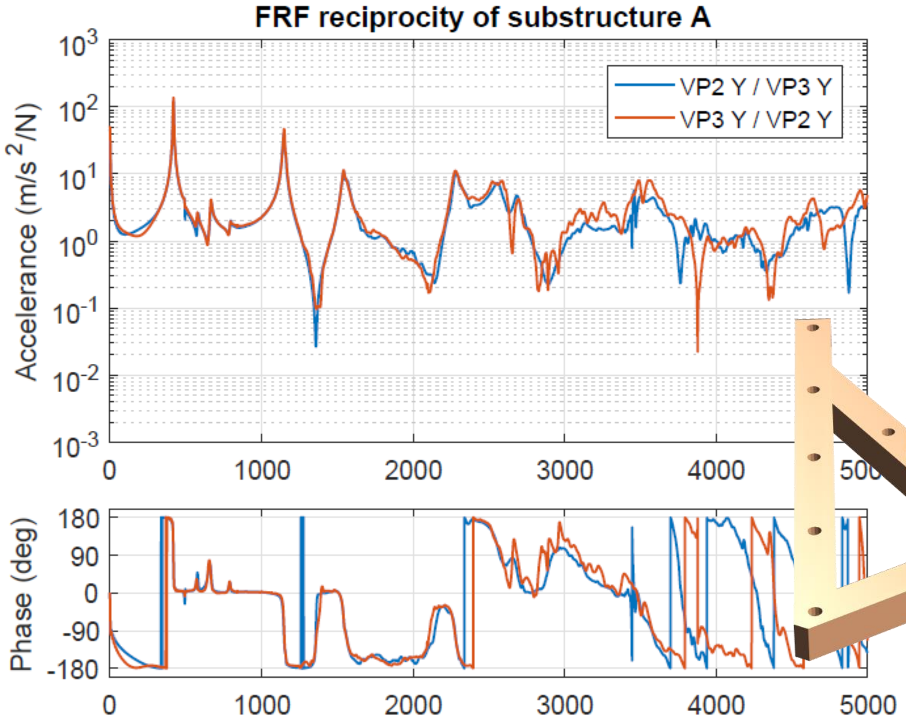


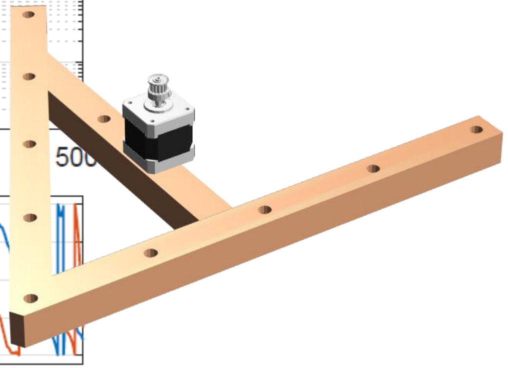
Examples

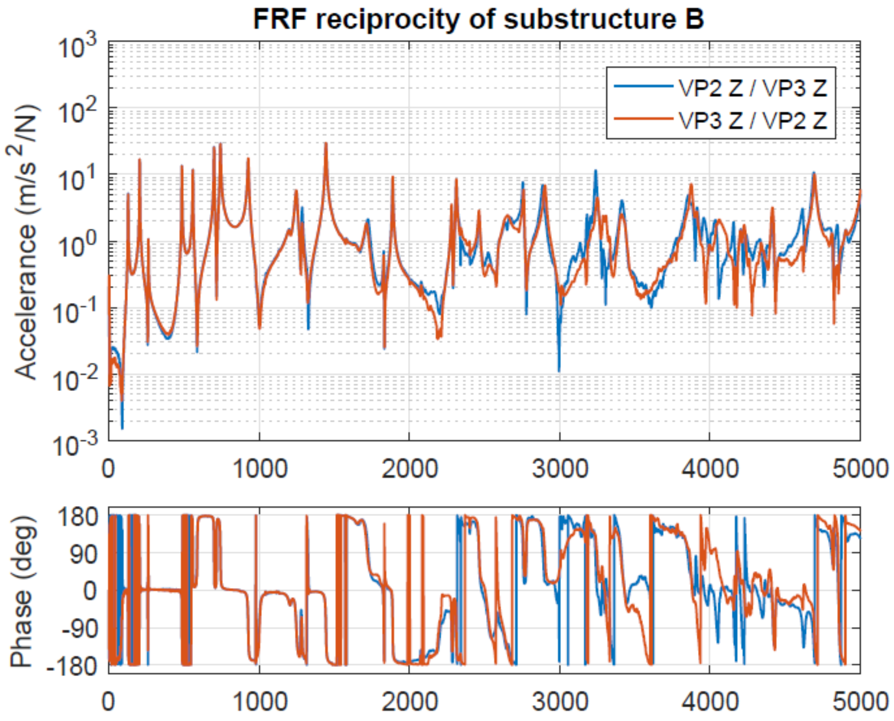
Substructure coupling using 2 experimentally modelled components



Results of measurement + virtual point transformation

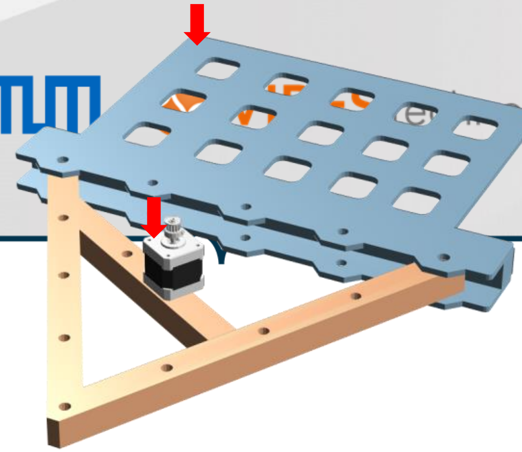




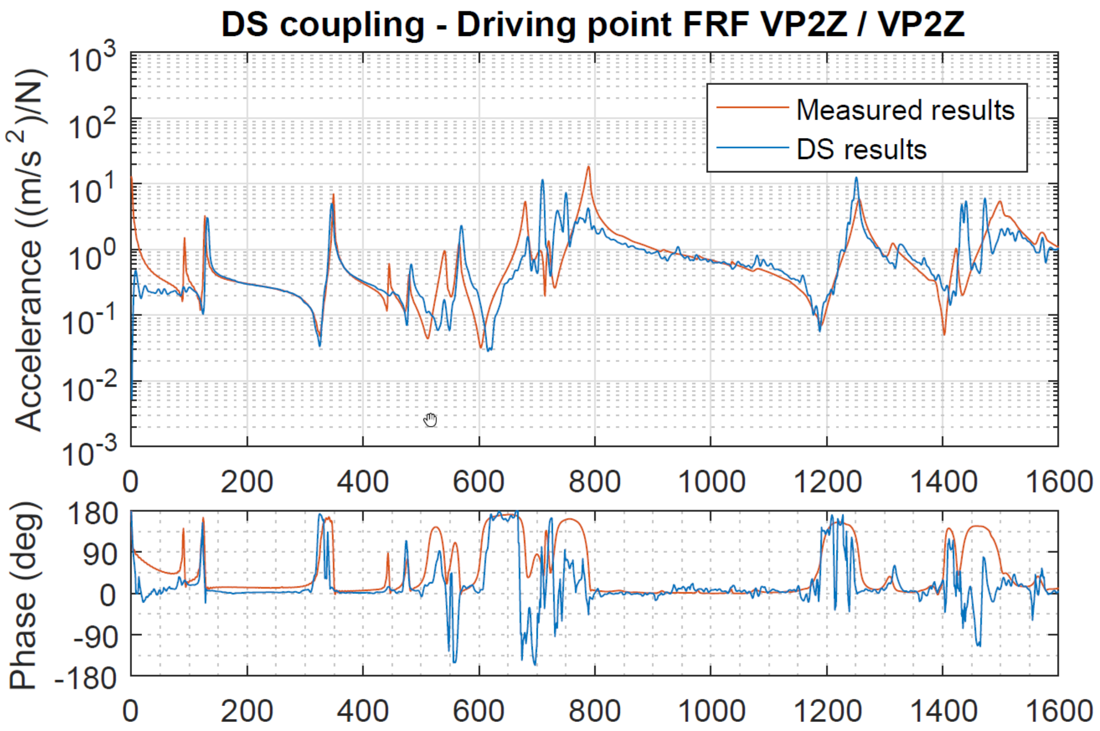


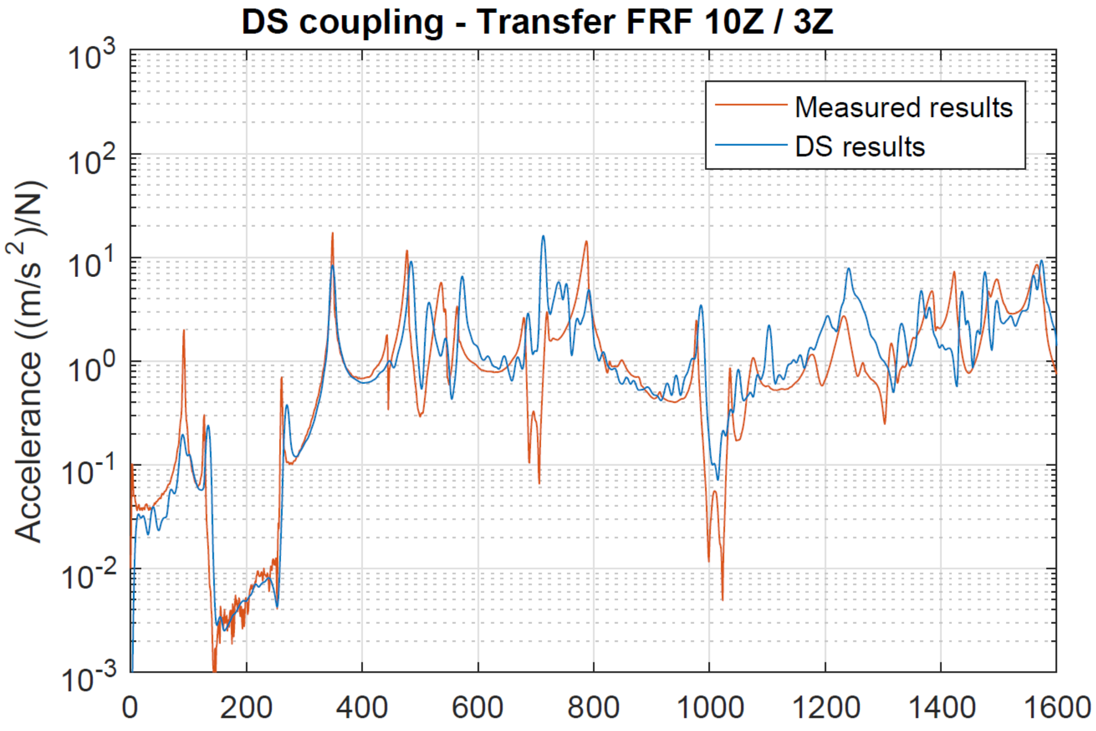
Examples

Substructure coupling using 2 experimentally modelled components



Results of experimental FBS (2x6 DoFs)





This tutorial will cover...

- **Substructuring Basics**
  - Coupling conditions: B and L-Matrix
  - Primal Assembly / CMS
  - Dual Assembly / LM-FBS

- **Practice of Dynamic Substructuring**
  - Virtual Point Transformation
  - Example: Experimental VP & FBS

- **Joint Identification**
  - Inverse substructuring
  - FBS decoupling
  - Example: Inverse substructuring of rubber mount